Comments on MR-2022-16 "Multidimensional encoding of restricted....."

A review by Thomas M Barbara, AIRC, OHSU, Portland, OR.

This effort is the latest in a long standing and ongoing program from the Lund group on diffusion as studied by magnetic resonance. In agreeing to referee this paper I wanted to finally study this topic more than a casual glance given in the past. Even though I have read both of the monographs by Callaghan I do not consider myself to be an expert in this field.

Extending past work to gradient modulations that mimic double rotation used in solid state NMR appears to be something with promising potential for advancing the field. However, I did not find much motivation for this in the submission. DOR in solid state NMR of course has strong motivation because of the need to average out both of Legendre's second and fourth order polynomials. If something like that exists in the diffusion field, that aspect did not receive proper attention in this submission.

I also find myself in something of a conundrum regarding the theoretical exposition of this paper. This has to do with my understanding of the "b matrix" as a "rank one" matrix and therefore such a matrix will have very simple eigenvalues, namely (0,0,1) apart from a scaling factor (See attached notes). If such an understanding is correct, then the b_sub_delta (equation 8) will have a very simple expression and the b_sub_eta (equation 10) will be zero for any modulation and for any frequency spectrum of such modulation. Therefore Equation 31 of this submission seems to not be connected at all with Equation 4 and readers such as myself will be left in the dark as to what exactly is going on. From this perspective the submission is significantly lacking in clarity. Such clarity should be exposed on a general level, without requiring a reader to get into the details of this or that construction of a specific modulation scheme.

It could turn out that the conundrum with rank one matrices is a simple misinterpretation of notation, and I would be happy if that was the case, but I feel sufficiently confident to bring it up in print. The resolution will likely require a significant revision if that is the case.

Tom Barbara

- The matrix b has elements

$$b_{ij} = g_i(\omega)\, g_j(-\omega) = g_i\, g_j^*$$

Thus b is a rank one matrix and

$$b^2 = |\underset{\sim}{g}(\omega)|^2\, b = \left\{\sum_k g_k(\omega)\, g_k^*(\omega)\right\} b$$

- Trace $b = |g|^2$ $\quad \mathrm{Tr}\, b^2 = |g|^4 = (\mathrm{Tr}\, b)^2$

The characteristic equation for the b matrix is therefore

$$\lambda^3 - (\mathrm{Tr}\, b)\,\lambda^2 + \overset{\overset{0}{\|}}{\tfrac{1}{2}\left((\mathrm{Tr}\, b)^2 - \mathrm{Tr}\, b^2\right)\lambda} - \overset{\overset{0}{\|}}{\det b} = 0$$

OR $\qquad \lambda^2\,(\lambda - |g|^2) = 0$

thus $\qquad b_\Delta = \dfrac{b_{zz}}{b} \qquad b_\eta = 0$