Fourier Transforms and Helmholtz

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$$\mathbf{H} = \int d^3\mathbf{r}' \; \boldsymbol{\nabla} \left[\mathbf{M}(\mathbf{r}') \cdot \boldsymbol{\nabla} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right] = \boldsymbol{\nabla} \Phi$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{2\pi^2} \int d^3\mathbf{k} \; \frac{1}{k^2} \, e^{i \mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\Phi = \frac{1}{2\pi^2} \int d^3\mathbf{k} \; i \, \frac{\hat{\mathbf{M}}(k)\cdot\mathbf{k}}{k^2} \, e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{H} = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \left\{ -4\pi \, \frac{\mathbf{k}\,(\mathbf{k}\cdot\hat{\mathbf{M}}(k))}{k^2} \right\} e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$= -\frac{4\pi}{3}\mathbf{M}(\mathbf{r}) + \int_{\mathbf{r}\neq\mathbf{r}'} d^3\mathbf{r}' \; \frac{3\left[\mathbf{M}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')\right](\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^5} - \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\mathbf{B} = -\boldsymbol{\nabla} \times \int d^3\mathbf{r}' \; \mathbf{M}(\mathbf{r}') \times \boldsymbol{\nabla} \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{(2\pi)^3} \int d^3\mathbf{r}' \; 4\pi \, \frac{\mathbf{k}\times(\hat{\mathbf{M}}\times\mathbf{k})}{k^2} \, e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$= \frac{1}{(2\pi)^3} \int d^3\mathbf{r}' \; 4\pi \left\{ \mathbf{M} - \frac{\mathbf{k}\,(\mathbf{k}\cdot\hat{\mathbf{M}}(k))}{k^2} \right\} e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\mathbf{H} + \frac{4\pi}{3}\mathbf{M} = B - \frac{8\pi}{3} M = \int_{\mathbf{r}\neq r'} d^3\mathbf{r}' \left[\text{dipole} \right]$$

Axially Averaged Dipole Field

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- Dipole at origin

$$\frac{3(\underset{\sim}{M}\cdot\underset{\sim}{r})\underset{\sim}{r}}{r^5}-\frac{\underset{\sim}{M}}{r^3} = \frac{1}{r^3}\left(3(\underset{\sim}{M}\cdot\underset{\sim}{\hat{r}})\underset{\sim}{\hat{r}} - \underset{\sim}{M}\right)$$

M fixed and averge over φ with $\underset{\sim}{\hat{r}}(\varphi) = R_z(\varphi)\underset{\sim}{\hat{r}}(0)$

- $$\underset{\sim}{\hat{r}}(\varphi) = (\underset{\sim}{n}\cdot\underset{\sim}{\hat{r}})\underset{\sim}{n} + c\varphi\left(\underset{\sim}{\hat{r}} - \underset{\sim}{n}(\underset{\sim}{n}\cdot\underset{\sim}{r})\right) - s\varphi\left(\underset{\sim}{n}\times\underset{\sim}{r}\right)$$

- after <u>LOTS</u> of vector algebra

for $\underset{\sim}{n} = \underset{\sim}{e}_z$!

$$\langle 3(\underset{\sim}{M}\cdot\underset{\sim}{\hat{r}})\underset{\sim}{\hat{r}} - \underset{\sim}{M}\rangle = \frac{\frac{1}{2}(3\hat{r}_z^2-1)}{r^3}\left\{3M_z\underset{\sim}{e}_z - \underset{\sim}{M}\right\}$$

- an easier method

$$E = \frac{1}{r^3}\left\{3(\underset{\sim}{M}\cdot\underset{\sim}{\hat{r}})(\underset{\sim}{N}\cdot\underset{\sim}{\hat{r}}) - \underset{\sim}{M}\cdot\underset{\sim}{N}\right\}$$

$$= \frac{1}{r^3}\left\{M_i\,(3\hat{r}_i\hat{r}_j - \delta_{ij})\,N_j\right\}$$

tensor notation

$$= \frac{1}{r^3}M_i D_{ij} N_j$$

- The energy is an invariant, but we can break the symmetry and transform only D_{ij}

$$\langle R_z(\varphi)\,D_{ij}\,R_z(-\varphi)\rangle = \begin{pmatrix} \frac{3}{2}(\hat{r}_x^2+\hat{r}_y^2)-1 & 0 & 0 \\ 0 & \frac{3}{2}(\hat{r}_x^2+\hat{r}_y^2)-1 & 0 \\ 0 & 0 & 3\hat{r}_z^2-1 \end{pmatrix}$$

- Again, this produces

$$\tfrac{1}{2}\left(3\hat{r}_z^2-1\right)\left\{3M_zN_z-(\underset{\sim}{M}\cdot\underset{\sim}{N})\right\}$$

for a general averaging about an axis $\underset{\sim}{n}$

$$\tfrac{1}{2}\left(3(\hat{\underset{\sim}{r}}\cdot\underset{\sim}{n})^2-1\right)\left\{3(\underset{\sim}{M}\cdot\underset{\sim}{n})\underset{\sim}{n}-\underset{\sim}{M}\right\} \quad \star$$

- NB we could also start with

$$\hat{r}_i\left(3M_iN_k-(\underset{\sim}{M}\cdot\underset{\sim}{N})\delta_{ik}\right)r_k$$

and apply the same trick.

- We can now apply this to the Fourier transform pair

$$\frac{1}{(2\pi)^3}\int d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, 4\pi\left\{\frac{(\underset{\sim}{k}\cdot\hat{\underset{\sim}{M}}(\underset{\sim}{k}))\underset{\sim}{k}}{\underset{\sim}{k}\cdot\underset{\sim}{k}}\right\} = \frac{4\pi}{3}\underset{\sim}{M}(\underset{\sim}{r})$$

$$+\int d^3\underset{\sim}{r}'\left\{\frac{\underset{\sim}{M}(\underset{\sim}{r}')}{|\underset{\sim}{r}-\underset{\sim}{r}'|^3}-\frac{3(\underset{\sim}{r}-\underset{\sim}{r}')\left[\underset{\sim}{M}(\underset{\sim}{r}')\cdot(\underset{\sim}{r}-\underset{\sim}{r}')\right]}{|\underset{\sim}{r}-\underset{\sim}{r}'|^5}\right\}$$

to obtain

$$\frac{1}{(2\pi)^3}\int d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\,\frac{4\pi}{3}\,\tfrac{1}{2}\left(3\frac{k_z^2}{k^2}-1\right)\left(3\hat{M}_z(k)\,\underset{\sim}{e}_z-\hat{\underset{\sim}{M}}(k)\right)$$

$$=\int d^3\underset{\sim}{r}'\,\frac{-\tfrac{1}{2}(3r_z^2-1)}{|\underset{\sim}{r}-\underset{\sim}{r}'|^3}\left\{3M_z(\underset{\sim}{r}')\,\underset{\sim}{e}_z-\underset{\sim}{M}(\underset{\sim}{r}')\right\}$$

$\star$ This trick can also be used for the expression in k space